FRICTIONLESS ASSET ALLOCATION WITH ELLIPTICALLY SYMMETRIC DISTRIBUTIONS OF RETURNS

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ABSTRACT. Frictionless asset allocation is examined for constant absolute risk aversion. The optimal portfolio for a general class of multivariate probability distributions with elliptical symmetry is presented. The optimal portfolio under these conditions is given by the portfolio obtained by mean-variance analysis scaled by a distribution dependent function. This result then specialized to the case of a multivariate extension of the generalized error distribution to allow the general properties of the solution to be exhibited.

1. UTILITY THEORY FORMALISM

1.1. **Asset Allocation Under Uncertainty.** Given the utility hypothesis, the approach to an asset allocation problem is straightforward. An agent purchases a portfolio, $\boldsymbol{C}$, of risky assets available for investment. Let the intertemporal change in the asset prices be $\boldsymbol{p}$, and the rate of return on riskless assets be r. With an initial position, $\boldsymbol{C}_0$, trading costs, $\boldsymbol{k}$, and margin payments, $\boldsymbol{m}$, we have:

$$W_2 = (1+r)W_1 - (1+r)\max(\boldsymbol{C}, -\boldsymbol{C})\cdot\boldsymbol{m} + \boldsymbol{C}\cdot\boldsymbol{p} - \max(\boldsymbol{C}-\boldsymbol{C}_0, \boldsymbol{C}_0-\boldsymbol{C})\cdot\boldsymbol{k}\,. \tag{1}$$

(In the above the vector $\boldsymbol{c} = \max(\boldsymbol{a}, \boldsymbol{b})$ is interpreted to be the result of the elementary operation $c_i = \max(a_i, b_i)$.) This expression has the form of a skewed convex hull. The optimal choice of investment, $\hat{\boldsymbol{C}}$, is the holding that maximizes the expected future utility. i.e.

$$\hat{\boldsymbol{C}} = \arg\max_{\boldsymbol{C}} E_{t_1} U\{W_2(\boldsymbol{C})\}. \tag{2}$$

1.2. **Negative Exponential Utility and Frictionless Trading.** For constant absolute risk aversion the utility function takes on the Pratt form $U(W) = -e^{-\lambda W}$. With negligible trading costs, this problem separates nicely to give

$$\hat{\boldsymbol{C}} = \arg\min_{\boldsymbol{C}} E_{t_1} e^{-\lambda \boldsymbol{C}\cdot\boldsymbol{p}}. \tag{3}$$

We recognize this as the moment generating function for the p.d.f. of $\boldsymbol{p}$.

2. ELLIPSOIDAL DISTRIBUTIONS

2.1. **General Considerations.** In this section we specialize our discussion to probability distributions with ellipsoidal symmetry. This is the set of continuous multivariate distributions that are constructed from a normalized symmetrical univariate distribution $f(x^2)$ by the substitution $\{x \to \boldsymbol{x}, f(x^2) \to \mathcal{A}f(g^2)\}$, where g is the Mahanalobis distance $\Delta_\Sigma(\boldsymbol{x}, \boldsymbol{\mu}) = \sqrt{(\boldsymbol{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$ and $\mathcal{A}$ is a constant introduced to normalize the constructed distribution over $\mathbb{R}^n$. These distributions are discussed extensively in reference [2].

Date: August 17, 2005. *Giller Investments Research Note*: 20040412/1.

2.1.1. *The Scaling Functions.* It can be shown that the m.g.f. of such a distribution is proportional to the function[2]

$$\psi_{\frac{n}{2}}(\boldsymbol{k}) = e^{-\boldsymbol{k}\cdot\boldsymbol{\mu}}\{\Delta(\boldsymbol{k})\}^{1-\frac{n}{2}} \int_0^\infty f(g^2) I_{\frac{n}{2}-1}\{g\Delta(\boldsymbol{k})\}\, g^{\frac{n}{2}}\, dg, \tag{4}$$

where $\boldsymbol{k} = \lambda\boldsymbol{C}$, $\Delta(\boldsymbol{k}) = \Delta_{\Sigma^{-1}}(\boldsymbol{k}, \boldsymbol{0}) = \sqrt{\boldsymbol{k}^T\Sigma\boldsymbol{k}}$ and n is the number of risky assets. ($I(\cdot)$ is the modified Bessel function of the first kind.) The root of the gradient of this function is shown to be the solution of

$$\lambda\boldsymbol{C}\Psi_{\frac{n}{2}}\{\lambda\Delta(\boldsymbol{C})\} = \Sigma^{-1}\boldsymbol{\mu} \text{ where } \Psi_\nu(x) = \frac{1}{x}\frac{\int_0^\infty f(g^2) I_\nu(gx)\, g^{\nu+1}\, dg}{\int_0^\infty f(g^2) I_{\nu-1}(gx)\, g^\nu\, dg}. \tag{5}$$

For the case of the multivariate normal distribution, the function $\Psi_\nu(x) = 1$ for all values of its arguments. The "scaling" function $\Psi_\nu(x)$ is a function that is sensitive to the behaviour of the p.d.f. in the tails and measures that behaviour relative to the normal.

The value of $\lambda\Delta(\boldsymbol{C})$ at the solution is the value $\hat{x}$ that solves

$$x\Psi_{\frac{n}{2}}(x) = \sqrt{\boldsymbol{\mu}^T\Sigma^{-1}\boldsymbol{\mu}}. \tag{6}$$

If it exists, we define the "inverting" function $\Phi_\nu(x)$ to be the root with respect to y of $y\Psi_\nu(y) = x$. Thus $\hat{x} = \Phi_\nu(\sqrt{\boldsymbol{\mu}^T\Sigma^{-1}\boldsymbol{\mu}})$ and the value to be used in Equation 5 is $\Psi_{\frac{n}{2}}\{\Phi_{\frac{n}{2}}(\sqrt{\boldsymbol{\mu}^T\Sigma^{-1}\boldsymbol{\mu}})\}$.

2.1.2. *The Optimal Portfolio.* The optimal portfolio is then given by

$$\hat{\boldsymbol{C}} = \frac{\Sigma^{-1}\boldsymbol{\mu}}{\lambda\Psi_{\frac{n}{2}}(\hat{x})}, \tag{7}$$

and the expected return on the portfolio is

$$\hat{\boldsymbol{C}}^T E_{t_1}\boldsymbol{x} = \frac{\boldsymbol{\mu}^T\Sigma^{-1}\boldsymbol{\mu}}{\lambda\Psi_{\frac{n}{2}}(\hat{x})}. \tag{8}$$

This expression has several interesting properties. Firstly, the optimal portfolio is always proportional to the portfolio $\Sigma^{-1}\boldsymbol{\mu}$, which is the solution to Markowitz's mean-variance optimization problem[**?**]. Secondly, the dependence on the parameter λ is a simple inverse scaling, which means that all investors with access to public information will be interested in obtaining the same portfolio in some proportion. i.e. A "market" portfolio can exist with these distributions and a C.A.P.M. style model will be constructable. Thirdly, the solution is completely independent of the wealth, W_1 (which is the result of specifying constant absolute risk aversion via the negative exponential utility function).

2.2. **The Generalized Error Distribution.**

2.2.1. *The Specific Form of the P.D.F.* Following the recipe of reference [2], we construct a multivariate generalized error distribution from the normalized univariate p.d.f. of the "parent" distribution. For the Generalized Error Distribution, we use

$$dF(x|\mu,\sigma,\kappa) = \left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}\right\}^{\frac{1}{2}} \frac{e^{-\left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}\left(\frac{x-\mu}{\sigma}\right)^2\right\}^{\frac{1}{2\kappa}}}}{2\sigma\Gamma(\kappa+1)}\, dx. \tag{9}$$

(This is not the usual manner in which the G.E.D. distribution is written[1]. In the usual form the variance of the p.d.f. is a strong function of κ, whereas in this form the univariate distribution is standardized for all κ.) For $\kappa = \frac{1}{2}$ Equation 9 represents the normal distribution; and, for $\kappa = 1$ the Laplace, or double exponential, distribution. The associated multivariate distribution is given by Equation 10, below.

(10)
$$dF(\boldsymbol{x}|\boldsymbol{\mu},\Sigma,\kappa) = \frac{d^n\boldsymbol{x}}{\sqrt{\pi^n|\Sigma|}}\frac{\Gamma(1+\frac{n}{2})}{\Gamma(1+n\kappa)}\left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}\right\}^{\frac{n}{2}} \exp - \left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}(\boldsymbol{x}-\boldsymbol{\mu})^T\Sigma^{-1}(\boldsymbol{x}-\boldsymbol{\mu})\right\}^{\frac{1}{2\kappa}}.$$

Unlike the univariate case, *this* p.d.f. is not standardized for $(\boldsymbol{\mu},\Sigma) = (\boldsymbol{0}, I)$, as the covariance matrix for $\boldsymbol{x}$ is given by

(11)
$$V = E\{(\boldsymbol{x}-\boldsymbol{\mu})(\boldsymbol{x}-\boldsymbol{\mu})^T\} = \xi^2(n,\kappa)\Sigma, \text{ where } \xi^2(n,\kappa) = \frac{\Gamma\{(2+n)\kappa\}\Gamma(1+\kappa)}{\Gamma(3\kappa)\Gamma(1+n\kappa)},$$

and this is a strongly increasing function of κ when $n > 1$. In particular, this means that $\Delta_\Sigma(\boldsymbol{x},\boldsymbol{y}) = \xi(n,\kappa)\Delta_V(\boldsymbol{x},\boldsymbol{y})$.

2.2.2. *Calculation of the Scaling Functions for a Single Asset.* Using the definition of Equation 9, we may write down an explicit form for the scaling function of Equation 4:

$$\Psi_\nu(x) = \frac{1}{x}\frac{\int_0^\infty e^{-\alpha g^{\frac{1}{k}}} I_\nu(gx)g^{\nu+1}\,dg}{\int_0^\infty e^{-\alpha g^{\frac{1}{k}}} I_{\nu-1}(gx)g^{\nu}\,dg} \text{ where } \alpha = \left\{\frac{\Gamma(3\kappa)}{\Gamma(\kappa)}\right\}^{\frac{1}{2\kappa}}. \tag{12}$$

Both of the integrands in Equation 12 contain a modified Bessel function factor and this function converges to $e^{gx}/\sqrt{2\pi gx}$ for large gx[3]. The rate of convergence depends on the order, ν, of the Bessel function but is true for all orders. This means that this Bessel function factor generally leads to exponential divergence of the integral. However, the divergence may be controlled by the exponential term arising from the p.d.f. as this is a convergent factor. Specifically, if $gx - \alpha g^{1/\kappa} > 0$ then the integral will diverge exponentially and if this term is negative then the integral will converge exponentially. Therefore we can conclude that the integrals will converge for all $0 < \kappa < 1$ and will converge for $\kappa = 1$ (Laplace distribution) if $x < \sqrt{2}$, but will diverge otherwise. This behaviour is illustrated for $\nu = \frac{1}{2}$ in Figure 1.

The sharp divergence illustrated for the Laplace distribution as $x \to \sqrt{2}$ has practical consequences for the computation of the inverting function, $\Phi_\nu(x)$. For the "regular" distributions (i.e. $0 < \kappa < 1$) $\Phi_\nu(x)$ is an unbounded increasing function of x. As $\kappa \to 1$ the function converges towards the value for the Laplace distribution, but it is never bounded above. For $\kappa = 1$ the function possesses an asymptote to $\sqrt{2}$ and is bounded below that level. This function is illustrated in Figure 2, on page 5.

The optimal portfolio is computed by scaling the normal theory portfolio $\Sigma^{-1}\boldsymbol{\mu}$ by the factor $1/\Psi_{n/2}(\hat{x})$. This scaling factor is illustrated in Figure 3 on page 6, and shows that for $\kappa > \frac{1}{2}$ the optimal portfolio for is never as heavily invested as the normal theory portfolio and is progressively less invested as risk/reward metric ($x = \boldsymbol{\mu}^T\Sigma\boldsymbol{\mu}$) increases.

The reason for this *scaleback* is clearly shown by Figure 4 on page 7. Here the expected portfolio return (assuming $\lambda = 1$) is plotted as $\kappa \to 1$. We see that for the normal theory portfolio the expected return is a quadratically increasing function of the risk/reward metric leading to heavy bets on large expected relative returns. These bets then dominate the

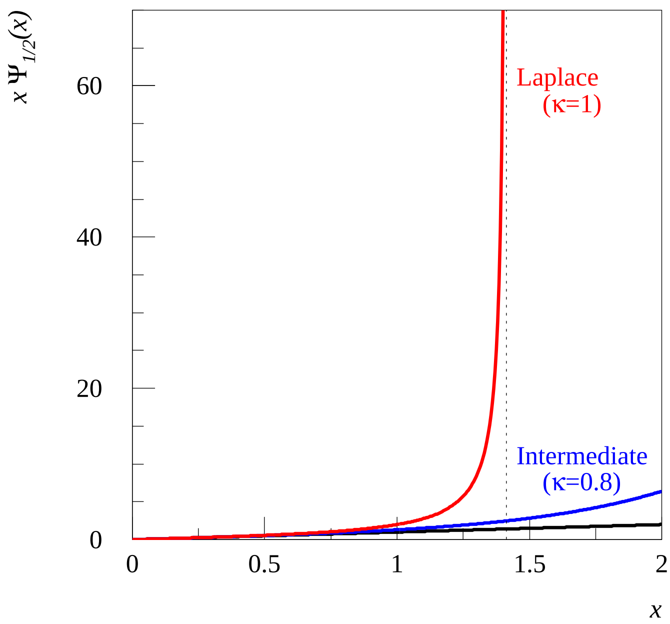

FIGURE 1. Behaviour of the scaling function $x\Psi_{1/2}(x)$ for $\kappa = 0.5, 0.8$ and 1.

profit stream from trading the asset. For non-normal theory these bets are dramatically curtailed, due to the progressively less "interesting"[1] nature of high risk/reward portfolios. We also see that a trader that implemented the normal theory portfolio in a more leptokurtotic market could be making a substantial overallocation of risk to reward and dramatically increasing their risk of ruin.

REFERENCES

[1] Evans, M., Hastings, N., & Peacock, B., "Statistical Distributions," 3rd. Edn., p. 74 John Wiley & Sons, Inc., 2000.

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[3] Gradshteyn, I.S. & Ryzhik, I.M., "Table of Integrals, Series, and Products," 6th. Edn., p. 909, Academic Press, 2000.

[4] Stuart, Alan & Ord, Keith, "Kendall's Advanced Theory of Statistics," Vol. I, 6th. Edn., p. 67, Edward Arnold, 1994.

[5] Merton, Robert C., "Continuous-time Finance," Revised Edn., p. 20, Blackwell, 1992.

[1]By "interesting" we are talking about the nominal statistical significance of the risk/reward metric. For the normal distribution ($\kappa = \frac{1}{2}$) a "5σ" expected return is very significant, and the trader's response is to make a heavy bet in those circumstances. For the Laplace distribution ($\kappa = 1$), such an expected return is much less significant and the trader in fact makes a smaller bet on the return.

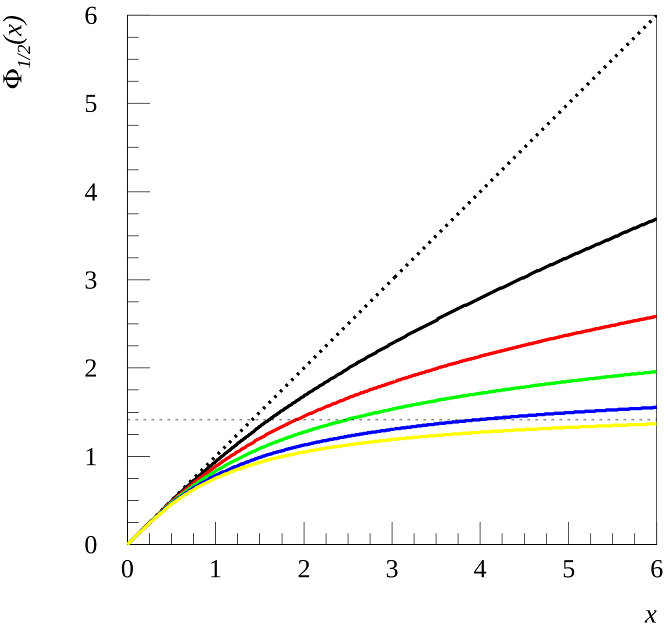

FIGURE 2. Behaviour of the Inverting Function $\Phi_{1/2}(x)$ as $\kappa \to 1$. The dotted diagonal line represents the normal distribution theory $\Phi_\nu(x) = 1$ and the dotted horizontal line shows the upper bound $\Phi_{1/2}(x) < \sqrt{2}$ for $\kappa = 1$.

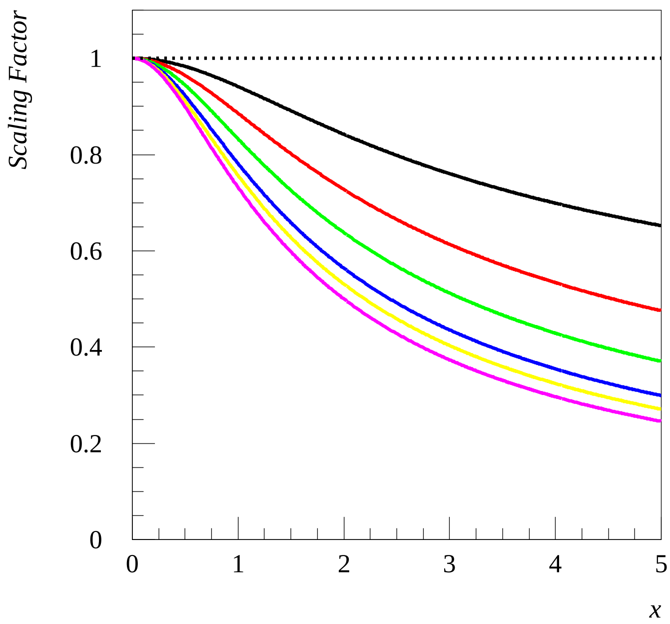

FIGURE 3. Portfolio scaling factors $1/\Psi_{1/2}\{\Phi_{\frac{1}{2}}(x)\}$ for a single asset as $\kappa \to 1$. The dotted line represents the normal distribution theory.

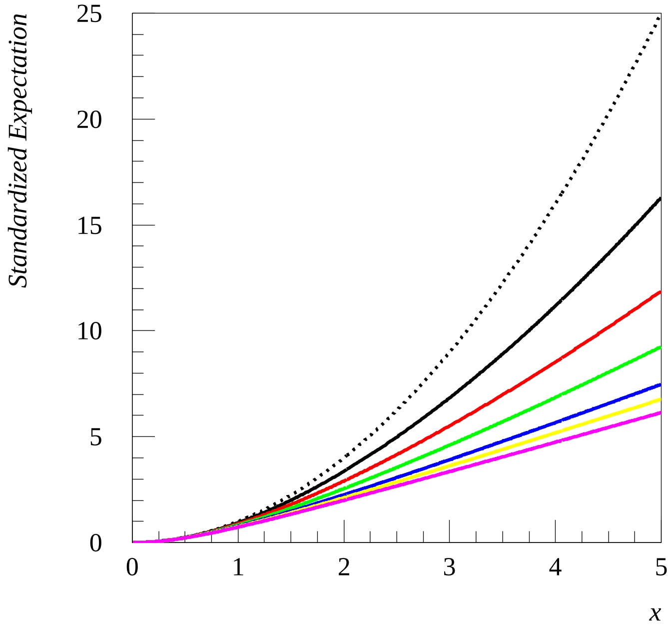

FIGURE 4. Standardized portfolio expected return $x^2/\Psi_{\frac{1}{2}}\{\Phi_{\frac{1}{2}}(x)\}$ for a single asset as $\kappa \to 1$. The dotted line represents the normal distribution theory.